J. Math. Anal. Appl. 393 (2012) 509–516

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Improved Hardy inequalities in the Grushin plane

Dan Su [a], Qiao-Hua Yang [b,*]

[a] School of Information Technology and Management Engineering, University of International Business and Economics, Beijing, 100029, People's Republic of China
[b] School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, People's Republic of China

ARTICLE INFO

Article history:
Received 2 January 2012
Available online 21 April 2012
Submitted by Manuel del Pino

Keywords:
Hardy inequality
Sobolev inequality
Grushin operator

ABSTRACT

We show the Hardy inequality for Grushin operator like $\partial_x^2 + 4x^2\partial_y^2$ on a bounded domain $\Omega \subset \mathbb{R}^2$ can be refined by adding remainder terms such as the improvement of Brezis–Vázquez, Vazquez–Zuazua and Filippas–Tertikas.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $0 \in \Omega$. The Hardy inequality reads, for all $f \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla f|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{f^2}{|x|^2} dx. \tag{1.1}$$

It is known that the constant $\frac{(N-2)^2}{4}$ in (1.1) is sharp and never archived. So, one could anticipate improving this inequality by adding some nonnegative correction term to the right-hand side of the inequality (1.1). In [1], Brezis and Vázquez have improved it by establishing that there exists a constant $C > 0$ such that for all $f \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla f|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{f^2}{|x|^2} dx + C \int_\Omega f^2 dx. \tag{1.2}$$

The constant C in (1.2) is given by $C = z_0^2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}}$, where ω_N and $|\Omega|$ denote the volume of the unit ball and Ω respectively, and $z_0 = 2.4048\cdots$ denotes the first zero of the Bessel function $J_0(z)$. Triggered by the work of Brezis and Vázquez, improved Hardy inequalities have been established in recent years by several authors. In particular, Vazquez and Zuazua [2] proved that for every $1 \leq q < 2$, there exists a constant $C(q, \Omega) > 0$, such that for all $f \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla f|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \frac{f^2}{|x|^2} dx + C(q,\Omega) \left(\int_\Omega |\nabla f|^q dx\right)^{\frac{2}{q}}. \tag{1.3}$$

Also motivated by the work of Brezis and Vázquez, Filippas and Tertikas [3, Theorem A], established the following Hardy–Sobolev inequality: for all $f \in C_0^\infty(\Omega)$, there holds, for some $C_N > 0$,

$$\int_\Omega |\nabla f|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{f^2}{|x|^2} dx \geq C_N \left(\int_\Omega X^{\frac{2(N-1)}{N-2}} \left(\frac{|x|}{D}\right) |f|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}, \tag{1.4}$$

* Correspondence to: L.M.B.A., Université de Bretagne-Sud, Centre de Recherche, Campus de Tohannic, BP 573, F-56017 Vannes, France.
E-mail addresses: sudan@uibe.edu.cn (D. Su), qhyang.math@gmail.com (Q.-H. Yang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.04.010

where $D > \sup_{x\in\Omega} |x|$ and

$$X(s) := (-\ln s)^{-1}, \quad 0 < s \le 1.$$

Furthermore, the best constant C_N in (1.4) can be exactly computed (for details, see [4]).

Our goal in this note is to establish analogous inequalities (1.2)–(1.4) for a Grushin operator like $\partial_x^2 + 4x^2\partial_y^2$. Recall that the Hardy inequality for $\partial_x^2 + 4x^2\partial_y^2$ states that, for all $f \in C_0^\infty(\mathbb{R}^2)$, there holds [5]

$$\int_{\mathbb{R}^2} |\nabla_L f|^2 dxdy \ge \frac{1}{4}\int_{\mathbb{R}^2} \frac{f^2}{\rho^2}|\nabla_L \rho|^2 dxdy \tag{1.5}$$

and $\frac{1}{4}$ is the best constant in (1.4), where $\nabla_L = (\partial_x, 2x\partial_y)$ and

$$\rho := \rho(x, y) = (x^4 + y^2)^{\frac{1}{4}}.$$

The Sobolev inequality on the Grushin plane reads that, for all $f \in C_0^\infty(\mathbb{R}^2)$, there holds (see [6,7])

$$\int_{\mathbb{R}^2} |\nabla_L f|^2 dxdy \ge \pi^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} |f|^6 dxdy\right)^{\frac{1}{3}}. \tag{1.6}$$

The inequality above is sharp and the extremal is given by $[(1+x^2)^2 + y^2]^{-1/4}$, up to a scaling and a vertical translation. For simplicity, we also denote by Ω a bounded domain in the Grushin plan $\mathbb{R}^2$ with $0 \in \Omega$. To this end we have:

Theorem 1.1 (*Brezis–Vázquez Improvement*)**.** *There exists a constant $C_1 > 0$ such that for all $f \in C_0^\infty(\Omega)$,*

$$\int_{\Omega} |\nabla_L f|^2 dxdy - \frac{1}{4}\int_{\Omega} \frac{f^2}{\rho^2}|\nabla_L \rho|^2 dxdy \ge C_1 \int_{\Omega} |f|^2 dxdy. \tag{1.7}$$

Theorem 1.2 (*Vazquez–Zuazua Improvement*)**.** *Let $1 \le q < 2$. There exists a constant $C_2 > 0$ such that for all $f \in C_0^\infty(\Omega)$,*

$$\int_{\Omega} |\nabla_L f|^2 dxdy - \frac{1}{4}\int_{\Omega} \frac{f^2}{\rho^2}|\nabla_L \rho|^2 dxdy \ge C_2 \left(\int_{\Omega} |\nabla_L f|^q dxdy\right)^{\frac{2}{q}}. \tag{1.8}$$

Theorem 1.3 (*Filippas–Tertikas Improvement*)**.** *There exists a constant $C_3 > 0$ such that for all $f \in C_0^\infty(\Omega)$,*

$$\int_{\Omega} |\nabla_L f|^2 dxdy - \frac{1}{4}\int_{\Omega} \frac{f^2}{\rho^2}|\nabla_L \rho|^2 dxdy \ge C_3 \left(\int_{\Omega} |f|^6 X^4\left(\frac{\rho(x,y)}{D}\right) dxdy\right)^{\frac{1}{3}}, \tag{1.9}$$

where $D > \sup_{(x,y)\in\Omega} \rho(x, y)$.

We note the main tools for the proof of inequalities (1.2)–(1.4) are either symmetric rearrangement [1] or the spherical harmonic decomposition [3,2]. However, to our knowledge, such rearrangement and decomposition do not exist for the Grushin operator. In order to prove the Theorems 1.1–1.3, we decompose the function in another way and this decomposition works (for details, see Section 3). This is the reason we can obtain inequalities (1.7)–(1.9).

We fail to compute the best constant in inequality (1.9). The main reason is that the extremal of inequality (1.6) is not radial. So the method used in [4] does not work.

2. Notations and preliminaries

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [6,5,8,7] for more precise information about the Grushin operator. Recall that the Grushin operator is the operator defined on $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$ by

$$\Delta_L = \frac{\partial^2}{\partial x^2} + 4x^2\frac{\partial^2}{\partial y^2}.$$

Δ_L is elliptic for $x \neq 0$ and degenerates on the manifold $\{0\} \times \mathbb{R}_y$. This operator belongs to the wide class of subelliptic operators studied by Franchi et al. in [8]. The operator Δ_L possesses a natural family of anisotropic dilations, namely

$$\delta_\lambda(x, y) = (\lambda x, \lambda^2 y), \quad \lambda > 0.$$

One easily checks that Δ_L is homogeneous of degree two with respect to $\{\delta_\lambda\}_{\lambda>0}$. The Jacobian of the dilations δ_λ is λ^Q, where $Q = 3$ is the homogeneous dimension. For simplicity, we will write $\lambda(x, y)$ to denote $\delta_\lambda(x, y)$. Denote by $\nabla_L = (\partial_x, 2x\partial_y)$. Then

$$\Delta_L = \langle \nabla_L, \nabla_L \rangle$$

and ∇_L is homogeneous of degree one with respect to $\{\delta_\lambda\}_{\lambda>0}$.

The anisotropic dilation structure on $\mathbb{R}^2$ introduces homogeneous norm

$$\rho := \rho(x, y) = (x^4 + y^2)^{\frac{1}{4}}.$$

A function f on $\mathbb{R}^2$ is said to be radial if $f(x, y) = \widetilde{f}(\rho)$. If f is radial, then (see [5])

$$|\nabla_L f| = \frac{|x|}{\rho}|f'(\rho)| \tag{2.1}$$

and

$$\Delta_L f = |\nabla_L \rho|^2 \left(f'' + \frac{2}{\rho}f'\right) = \frac{|x|^2}{\rho^2}\left(f'' + \frac{2}{\rho}f'\right). \tag{2.2}$$

With the norm ρ, we can define the ball centered at origin with radius R

$$B_R = \{(x, y) \in \mathbb{R}^2 : \rho(x, y) < R\}$$

and the unit sphere $\Sigma = \partial B_1 = \{(x, y) \in \mathbb{R}^2 : \rho(x, y) = 1\}$. Given any $(0, 0) \neq \xi = (x, y) \in \mathbb{R}^2$, set $x^* = \frac{x}{\rho(x,y)}, y^* = \frac{y}{\rho(x,y)^2}$ and $\xi^* = (x^*, y^*)$. The polar coordinates on $\mathbb{R}^2$ associated with ρ is the following (cf. [9, Proposition 1.15]):

$$\int_{\mathbb{R}^2} f(x, y)dxdy = \int_0^\infty \int_\Sigma f(\lambda(x^*, y^*))\lambda^2 d\sigma d\lambda, \quad f \in L^1(\mathbb{R}^2).$$

Next we give a parametrization of the Grushin plane and refer to [5, p. 728] and [10, Theorem 5.12] for more information about this parametrization. On $[0, +\infty) \times (-\infty, +\infty)$, we set

$$x = \rho\sqrt{\sin\theta}, \qquad y = \rho^2\cos\theta, \quad 0 \leq \theta \leq \pi, \ 0 < \rho < +\infty.$$

Then for all $f \in L^1([0, +\infty) \times (-\infty, +\infty))$,

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} f(x, y)dxdy = \int_0^{+\infty}\int_0^\pi f(x(\rho, \theta), y(\rho, \theta))\left(\sqrt{\sin\theta}\right)^{-1}\rho^2 d\rho d\theta. \tag{2.3}$$

On the other hand, it is easy to check that, for $f \in C_0^\infty([0, +\infty) \times (-\infty, +\infty))$,

$$\frac{\partial f}{\partial \rho} = \sqrt{\sin\theta}\frac{\partial f}{\partial x} + 2\rho\cos\theta\frac{\partial f}{\partial y}, \qquad \frac{\partial f}{\partial \theta} = \frac{\rho\cos\theta}{2\sqrt{\sin\theta}}\frac{\partial f}{\partial x} - \rho^2\sin\theta\frac{\partial f}{\partial y},$$

$$\sin\theta\left(\left|\frac{\partial f}{\partial \rho}\right|^2 + \frac{4}{\rho^2}\left|\frac{\partial f}{\partial \theta}\right|^2\right) = \left|\frac{\partial f}{\partial x}\right|^2 + 4x^2\left|\frac{\partial f}{\partial y}\right|^2 = |\nabla_L f|^2.$$

Therefore,

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} |\nabla_L f|^2 dxdy = \int_0^{+\infty}\int_0^\pi \left(\left|\frac{\partial f}{\partial \rho}\right|^2 + \frac{4}{\rho^2}\left|\frac{\partial f}{\partial \theta}\right|^2\right)\sqrt{\sin\theta}\rho^2 d\rho d\theta. \tag{2.4}$$

Similarly, on $(-\infty, 0] \times (-\infty, +\infty)$, we set

$$x = -\rho\sqrt{-\sin\theta}, \qquad y = \rho^2\cos\theta, \quad \pi \leq \theta \leq 2\pi, \ 0 < \rho < +\infty.$$

Then for all $f \in L^1((-\infty, 0] \times (-\infty, +\infty))$,

$$\int_{-\infty}^0\int_{-\infty}^{+\infty} f(x, y)dxdy = \int_0^{+\infty}\int_\pi^{2\pi} f(x(\rho, \theta), y(\rho, \theta))\left(\sqrt{-\sin\theta}\right)^{-1}\rho^2 d\rho d\theta \tag{2.5}$$

and for $f \in C_0^\infty((-\infty, 0] \times (-\infty, +\infty))$,

$$\int_{-\infty}^0\int_{-\infty}^{+\infty} |\nabla_L f|^2 dxdy = \int_0^{+\infty}\int_\pi^{2\pi} \left(\left|\frac{\partial f}{\partial \rho}\right|^2 + \frac{4}{\rho^2}\left|\frac{\partial f}{\partial \theta}\right|^2\right)\sqrt{-\sin\theta}\rho^2 d\rho d\theta. \tag{2.6}$$

Combining (2.3)–(2.6) yields, for $f \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} f(x, y)dxdy = \int_0^{+\infty}\int_0^{2\pi} f(x(\rho, \theta), y(\rho, \theta))\left(\sqrt{|\sin\theta|}\right)^{-1}\rho^2 d\rho d\theta \tag{2.7}$$

and

$$\int_{\mathbb{R}^2} |\nabla_L f|^2 dxdy = \int_0^{+\infty}\int_0^{2\pi} \left(\left|\frac{\partial f}{\partial \rho}\right|^2 + \frac{4}{\rho^2}\left|\frac{\partial f}{\partial \theta}\right|^2\right)\sqrt{|\sin\theta|}\rho^2 d\rho d\theta. \tag{2.8}$$

Set, for $\gamma > -1$,

$$A_\gamma := \int_\Sigma |\nabla_L \rho|^\gamma d\sigma = \int_\Sigma |x^*|^\gamma d\sigma = \int_0^{2\pi} |\sin\theta|^{\frac{\gamma-1}{2}} d\theta. \tag{2.9}$$

Closely following [11, pp. 1572–1573], we have

$$\begin{aligned} A_\gamma &= (3+\gamma)\int_0^1 \rho^{\gamma+2} d\rho \int_\Sigma |x^*|^\gamma d\sigma = (3+\gamma)\int_\Sigma \int_0^1 |\rho x^*|^\gamma \rho^2 d\rho d\sigma \\ &= (3+\gamma)\int_{\rho<1} |x|^\gamma dxdy = (3+\gamma)\int_{|y|<1}\int_{|x|<(1-|y|^2)^{\frac14}} |x|^\gamma dxdy \\ &= (3+\gamma)\frac{2}{1+\gamma}\int_{|y|<1} (1-|y|^2)^{\frac{\gamma+1}{4}} dy = \frac{4(3+\gamma)}{1+\gamma}\int_0^1 (1-y^2)^{\frac{\gamma+1}{4}} dy \\ &= \frac{4(3+\gamma)}{1+\gamma}\cdot\frac12\int_0^1 (1-t)^{\frac{\gamma+1}{4}} t^{-\frac12} dt \\ &= \frac{2(3+\gamma)}{1+\gamma}\frac{\Gamma\left(\frac{\gamma+5}{4}\right)\Gamma(1/2)}{\Gamma\left(\frac{\gamma+7}{4}\right)} = 2\frac{\Gamma\left(\frac{\gamma+1}{4}\right)\Gamma(1/2)}{\Gamma\left(\frac{\gamma+3}{4}\right)} < +\infty. \end{aligned} \tag{2.10}$$

3. The proofs

For every $f \in C_0^\infty(\mathbb{R}^2)$, we decompose f into

$$f(x,y) = f_1(\rho(x,y)) + f_2(x,y),$$

where $f_1(\rho)$ is the integral mean of f over the unit sphere Σ with weight $|\nabla_L\rho|^2$, that is

$$f_1(\rho) := \frac{1}{\int_\Sigma |\nabla_L\rho|^2 d\sigma}\int_\Sigma f|\nabla_L\rho|^2 d\sigma.$$

Then $f_1, f_2 \in C_0^\infty(\mathbb{R}^2)$. Furthermore, this decomposition satisfies the following:

Lemma 3.1. *Let $f \in C_0^\infty(\mathbb{R}^2)$. There holds,*
(1) *for $\alpha < 3$,*

$$\int_{\mathbb{R}^2} \frac{|f|^2}{\rho^\alpha}|\nabla_L\rho|^2 dxdy = \int_{\mathbb{R}^2} \frac{|f_1|^2}{\rho^\alpha}|\nabla_L\rho|^2 dxdy + \int_{\mathbb{R}^2} \frac{|f_2|^2}{\rho^\alpha}|\nabla_L\rho|^2 dxdy;$$

(2) $\int_{\mathbb{R}^2} |\nabla_L f|^2 dxdy = \int_{\mathbb{R}^2} |\nabla_L f_1|^2 dxdy + \int_{\mathbb{R}^2} |\nabla_L f_2|^2 dxdy$.

Proof. (1) Notice that

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|f_2|^2}{\rho^\alpha}|\nabla_L\rho|^2 &= \int_{\mathbb{R}^2} \frac{|f-f_1|^2}{\rho^\alpha}|\nabla_L\rho|^2 \\ &= \int_{\mathbb{R}^2} \frac{|f|^2}{\rho^\alpha}|\nabla_L\rho|^2 + \int_{\mathbb{R}^2} \frac{|f_1|^2}{\rho^\alpha}|\nabla_L\rho|^2 - 2\int_{\mathbb{R}^2} \frac{ff_1}{\rho^\alpha}|\nabla_L\rho|^2. \end{aligned}$$

Using polar coordinates, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{ff_1}{\rho^\alpha}|\nabla_L\rho|^2 &= \int_0^{+\infty} f_1(\rho)\rho^{2-\alpha}\cdot\left(\int_\Sigma f|\nabla_L\rho|^2 d\sigma\right)d\rho \\ &= \int_0^{+\infty} f_1(\rho)\rho^{2-\alpha}\cdot f_1(\rho)\int_\Sigma |\nabla_L\rho|^2 d\sigma d\rho \\ &= \int_{\mathbb{R}^2} \frac{|f_1|^2}{\rho^\alpha}|\nabla_L\rho|^2 dxdy. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|f_2|^2}{\rho^\alpha}|\nabla_L\rho|^2 &= \int_{\mathbb{R}^2} \frac{|f|^2}{\rho^\alpha}|\nabla_L\rho|^2 + \int_{\mathbb{R}^2} \frac{|f_1|^2}{\rho^\alpha}|\nabla_L\rho|^2 - 2\int_{\mathbb{R}^2} \frac{ff_1}{\rho^\alpha}|\nabla_L\rho|^2 \\ &= \int_{\mathbb{R}^2} \frac{|f|^2}{\rho^\alpha}|\nabla_L\rho|^2 - \int_{\mathbb{R}^2} \frac{|f_1|^2}{\rho^\alpha}|\nabla_L\rho|^2. \end{aligned}$$

The desired result follows.

(2) The proof is similar to that of (1). Using (2.2), we have

$$\begin{aligned}\int_{\mathbb{R}^2}\langle\nabla_L f,\nabla_L f_1\rangle &= -\int_{\mathbb{R}^2} f\Delta_L f_1 = -\int_0^{+\infty}\int_\Sigma f|\nabla_L\rho|^2\left(f_1''+\frac{2}{\rho}f_1'\right)\rho^2 d\sigma\, d\rho\\ &= -\int_0^{+\infty}\left(f_1''+\frac{2}{\rho}f_1'\right)\rho^2\cdot\left(\int_\Sigma f|\nabla_L\rho|^2 d\sigma\right)d\rho\\ &= -\int_0^{+\infty}\left(f_1''+\frac{2}{\rho}f_1'\right)\rho^2\cdot f_1\int_\Sigma |\nabla_L\rho|^2 d\sigma\, d\rho\\ &= -\int_0^{+\infty}\int_\Sigma f_1\left(f_1''+\frac{2}{\rho}f_1'\right)|\nabla_L\rho|^2\rho^2 d\sigma\, d\rho\\ &= -\int_{\mathbb{R}^2} f_1\Delta_L f_1 = \int_{\mathbb{R}^2}|\nabla_L f_1|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla_L f_2|^2 &= \int_{\mathbb{R}^2}|\nabla_L f-\nabla_L f_1|^2\\ &= \int_{\mathbb{R}^2}|\nabla_L f|^2+\int_{\mathbb{R}^2}|\nabla_L f_1|^2-2\int_{\mathbb{R}^2}\langle\nabla_L f,\nabla_L f_1\rangle\\ &= \int_{\mathbb{R}^2}|\nabla_L f|^2-\int_{\mathbb{R}^2}|\nabla_L f_1|^2.\end{aligned}$$

The proof of Lemma 3.1 is now completed. □

Lemma 3.2. *There exists a constant $C>0$, such that*

$$\int_{\mathbb{R}^2}|\nabla_L f_2|^2 dxdy-\frac{1}{4}\int_{\mathbb{R}^2}\frac{|f_2|^2}{\rho^2}|\nabla_L\rho|^2 dxdy \ge C\int_{\mathbb{R}^2}|\nabla_L f_2|^2 dxdy.$$

Proof. Set

$$\mu=\inf\left\{\frac{\int_0^{2\pi}|u'(\theta)|^2\sqrt{|\sin\theta|}d\theta}{\int_0^{2\pi}|u(\theta)|^2\sqrt{|\sin\theta|}d\theta} : u\in C^1([0,2\pi]),\ \int_0^{2\pi}u(\theta)\sqrt{|\sin\theta|}d\theta=0, u\neq 0\right\}.$$

Since $\int_0^{2\pi} f_2\sqrt{|\sin\theta|}d\theta=\int_\Sigma f_2|\nabla_L\rho|^2 d\sigma=\int_\Sigma f|\nabla_L\rho|^2 d\sigma - f_1\int_\Sigma|\nabla_L\rho|^2 d\sigma=0$, we have, by (2.8),

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla_L f_2|^2 dxdy &= \int_0^{+\infty}\int_0^{2\pi}\left(\left|\frac{\partial f_2}{\partial\rho}\right|^2+\frac{4}{\rho^2}\left|\frac{\partial f_2}{\partial\theta}\right|^2\right)\sqrt{|\sin\theta|}\rho^2 d\rho d\theta\\ &\ge \int_0^{+\infty}\int_0^{2\pi}\left(\left|\frac{\partial f_2}{\partial\rho}\right|^2+\frac{4\mu|f_2|^2}{\rho^2}\right)\sqrt{|\sin\theta|}\rho^2 d\rho d\theta\\ &= \int_0^{+\infty}\int_0^{2\pi}\left(\left|\frac{\partial f_2}{\partial\rho}\right|^2\rho^2+4\mu|f_2|^2\right)\sqrt{|\sin\theta|}d\rho d\theta.\end{aligned}\tag{3.1}$$

On the other hand,

$$\begin{aligned}&\int_0^{+\infty}\int_0^{2\pi}\left|\frac{\partial f_2}{\partial\rho}\right|^2\rho^2\sqrt{|\sin\theta|}d\rho d\theta-\frac{1}{4}\int_0^{+\infty}\int_0^{2\pi}|f_2|^2\sqrt{|\sin\theta|}d\rho d\theta\\ &= \int_0^{2\pi}\left(\int_0^{+\infty}\left|\frac{\partial f_2}{\partial\rho}\right|^2\rho^2 d\rho-\frac{1}{4}\int_0^{+\infty}|f_2|^2 d\rho\right)\sqrt{|\sin\theta|}d\theta\\ &= \int_0^{2\pi}\left(\int_0^{+\infty}\left|\frac{\partial\left(\sqrt{\rho}f_2\right)}{\partial\rho}\right|^2\rho d\rho\right)\sqrt{|\sin\theta|}d\theta\ge 0.\end{aligned}\tag{3.2}$$

Combining (3.1) and (3.2) yields

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla_L f_2|^2 dxdy &\ge \left(\frac{1}{4}+4\mu\right)\int_0^{+\infty}\int_0^{2\pi} |f_2|^2\sqrt{|\sin\theta|}d\rho d\theta \\ &= \left(\frac{1}{4}+4\mu\right)\int_{\mathbb{R}^2}\frac{|f_2|^2}{\rho^2}|\nabla_L\rho|^2 dxdy,\end{aligned}$$

i.e.

$$\int_{\mathbb{R}^2} |\nabla_L f_2|^2 dxdy - \frac{1}{4}\int_{\mathbb{R}^2}\frac{|f_2|^2}{\rho^2}|\nabla_L\rho|^2 dxdy \ge \frac{16\mu}{1+16\mu}\int_{\mathbb{R}^2}|\nabla_L f_2|^2 dxdy.$$

To finish the proof, it is enough to show $\mu > 0$. Let $u \in C^1([0, 2\pi])$ satisfying $\int_0^{2\pi} u(\theta)\sqrt{|\sin\theta|}d\theta = 0$. Then for each $\phi \in [0, 2\pi]$,

$$\begin{aligned}|u(\phi)| &= \left|u(\phi) - \frac{\int_0^{2\pi} u(\theta)\sqrt{|\sin\theta|}d\theta}{\int_0^{2\pi}\sqrt{|\sin\theta|}d\theta}\right| = \left|\frac{\int_0^{2\pi}(u(\phi)-u(\theta))\sqrt{|\sin\theta|}d\theta}{\int_0^{2\pi}\sqrt{|\sin\theta|}d\theta}\right| \\ &= \left|\frac{\int_0^{2\pi}\int_\theta^\phi u'(t)dt\sqrt{|\sin\theta|}d\theta}{\int_0^{2\pi}\sqrt{|\sin\theta|}d\theta}\right| \le \frac{\int_0^{2\pi}\left|\int_\theta^\phi u'(t)dt\right|\sqrt{|\sin\theta|}d\theta}{\int_0^{2\pi}\sqrt{|\sin\theta|}d\theta} \\ &\le \frac{\int_0^{2\pi}\int_0^{2\pi}|u'(t)|dt\sqrt{|\sin\theta|}d\theta}{\int_0^{2\pi}\sqrt{|\sin\theta|}d\theta} = \int_0^{2\pi}|u'(t)|dt \\ &\le \left(\int_0^{2\pi}|u'(t)|^2\sqrt{|\sin t|}dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{2\pi}\left(\sqrt{|\sin t|}\right)^{-1}dt\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\int_0^{2\pi}|u(\phi)|^2\sqrt{|\sin\phi|}d\phi \le \int_0^{2\pi}|u'(t)|^2\sqrt{|\sin t|}dt\cdot\int_0^{2\pi}\frac{dt}{\sqrt{|\sin t|}}\cdot\int_0^{2\pi}\sqrt{|\sin\phi|}d\phi.$$

We obtain, by (2.9) and (2.10),

$$\mu \ge \frac{1}{\int_0^{2\pi}\frac{dt}{\sqrt{|\sin t|}}\cdot\int_0^{2\pi}\sqrt{|\sin\phi|}d\phi} = \frac{1}{A_0A_1} > 0.$$

The desired result follows. □

Proof of Theorem 1.1. Without loss of generality, we assume $\Omega = B_R$ for some $R > 0$. In fact, if Ω is a bounded domain, then $\Omega \subset B_R$ with $R = \sup_{(x,y)\in\Omega}\rho(x, y)$. If Theorem 1.1 is true for any $u \in C_0^\infty(B_R)$, then it is also true for any $u \in C_0^\infty(\Omega)$.

Since f_1 is radial,

$$\int_{B_R}|\nabla_L f_1|^2 dxdy - \frac{1}{4}\int_{B_R}\frac{f_1^2}{\rho^2}|\nabla_L\rho|^2 dxdy = A_2\left(\int_0^R |f_1'|^2\rho^2 d\rho - \tfrac{1}{4}\int_0^R |f_1|^2 d\rho\right),$$

where A_2 is defined in (2.9). Following [1], we have, for some $C > 0$,

$$\int_0^R |f_1'|^2\rho^2 d\rho - \frac{1}{4}\int_0^R |f_1|^2 d\rho = \int_0^R\left(\sqrt{\rho}f_1\right)'\rho d\rho \ge C\int_0^R |f_1|^2\rho^2 d\rho.$$

Therefore,

$$\int_{B_R}|\nabla_L f_1|^2 dxdy - \frac{1}{4}\int_{B_R}\frac{f_1^2}{\rho^2}|\nabla_L\rho|^2 dxdy \ge CA_2\int_0^R |f_1|^2\rho^2 d\rho = \frac{CA_2}{A_0}\int_{B_R}|f_1|^2 dxdy,$$

where A_0 is defined in (2.9). On the other hand, by Lemma 3.2,

$$\int_{B_R}|\nabla_L f_2|^2 dxdy - \frac{1}{4}\int_{B_R}\frac{|f_2|^2}{\rho^2}|\nabla_L\rho|^2 dxdy \ge C\int_{B_R}|\nabla_L f_2|^2 dxdy \ge \frac{C}{4R^2}\int_{B_R}|f_2|^2 dxdy.$$

To get the last inequality above, we use the Poincaré inequality (see [5, Theorem 3.7])

$$\int_{B_R}|\nabla_L f_2|^2 dxdy \ge \frac{1}{4R^2}\int_{B_R}|f_2|^2 dxdy.$$

It then follows from Lemma 3.1, for some $C, C' > 0$,

$$\begin{aligned}\int_{B_R} |\nabla_L f|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{f^2}{\rho^2}|\nabla_L \rho|^2 dxdy &= \int_{B_R} |\nabla_L f_1|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{f_1^2}{\rho^2}|\nabla_L \rho|^2 dxdy \\ &\quad + \int_{B_R} |\nabla_L f_2|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{f_2^2}{\rho^2}|\nabla_L \rho|^2 dxdy \\ &\geq C\int_{B_R} |f_1|^2 dxdy + C\int_{B_R} |f_2|^2 dxdy \geq C'\int_{B_R} |f|^2 dxdy.\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1. Without loss of generality, we may also assume $\Omega = B_R$. Since f_1 is radial,

$$\begin{aligned}\int_{B_R} |\nabla_L f_1|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{f_1^2}{\rho^2}|\nabla_L \rho|^2 dxdy &= A_2\left(\int_0^R |f_1'|^2\rho^2 d\rho - \frac{1}{4}\int_0^R |f_1|^2 d\rho\right) \\ &\geq C\left(\int_0^R |f_1'|^q\rho^2 d\rho\right)^{\frac{2}{q}} = \frac{C}{A_q^{\frac{1}{q}}}\left(\int_{B_R} |\nabla_L f_1|^q dxdy\right)^{\frac{2}{q}},\end{aligned} \tag{3.3}$$

where A_q is defined in (2.9). On the other hand, by Lemma 3.2,

$$\int_{B_R} |\nabla_L f_2|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{|f_2|^2}{\rho^2}|\nabla_L \rho|^2 dxdy \geq C\int_{B_R} |\nabla_L f_2|^2 dxdy \geq \frac{C}{|B_R|^{\frac{2-q}{q}}}\left(\int_{B_R} |\nabla_L f_2|^q dxdy\right)^{\frac{2}{q}}, \tag{3.4}$$

where $|B_R|$ is the volume of B_R. To get the last inequality above, we use the Hölder inequality:

$$\begin{aligned}\int_{B_R} |\nabla_L f_2|^q dxdy &\leq \left(\int_{B_R} |\nabla_L f_2|^2 dxdy\right)^{\frac{q}{2}} \cdot \left(\int_{B_R} dxdy\right)^{\frac{2-q}{2}} \\ &= |B_R|^{\frac{2-q}{2}}\left(\int_{B_R} |\nabla_L f_2|^2 dxdy\right)^{\frac{q}{2}}.\end{aligned}$$

It remains to combine (3.3) and (3.4) and the desired result follows. □

Proof of Theorem 1.3. The proof is similar to that of Theorems 1.1 and 1.2. As in the proof of Theorems 1.1 and 1.2, we can consider only the case $\Omega = B_R$. Using the polar coordinates, one can easily check that Theorem 1.3 is true for radial functions. On the other hand, by Lemma 3.2, for some $C, C' > 0$,

$$\begin{aligned}\int_{B_R} |\nabla_L f_2|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{|f_2|^2}{\rho^2}|\nabla_L \rho|^2 dxdy &\geq C\int_{B_R} |\nabla_L f_2|^2 dxdy \\ &\geq C\pi^{\frac{2}{3}}\left(\int_{B_R} |f_2|^6 dxdy\right)^{\frac{1}{3}} \geq C'\left(\int_{B_R} |f_2|^6 X^4\left(\frac{\rho}{D}\right) dxdy\right)^{\frac{1}{3}}.\end{aligned}$$

To get the last inequality above, we use the fact that $X^4(\frac{\rho}{D})$ is bounded.

Therefore,

$$\begin{aligned}\int_{B_R} |\nabla_L f|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{f^2}{\rho^2}|\nabla_L \rho|^2 dxdy &= \int_{B_R} |\nabla_L f_1|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{f_1^2}{\rho^2}|\nabla_L \rho|^2 dxdy \\ &\quad + \int_{B_R} |\nabla_L f_2|^2 dxdy - \frac{1}{4}\int_{B_R} \frac{f_2^2}{\rho^2}|\nabla_L \rho|^2 dxdy \\ &\geq C'\left(\int_{B_R} |f_1|^6 X^4\left(\frac{\rho}{D}\right) dxdy\right)^{\frac{1}{3}} + C'\left(\int_{B_R} |f_2|^6 X^4\left(\frac{\rho}{D}\right) dxdy\right)^{\frac{1}{3}} \\ &\geq C''\left(\int_{B_R} |f|^6 X^4\left(\frac{\rho}{D}\right) dxdy\right)^{\frac{1}{3}}.\end{aligned}$$

The proof of Theorem 1.3 is now completed. □

Acknowledgments

The authors thanks the referee for his/her careful reading and very useful comments which improved the final version of this paper.

The first author is supported by Program for Research Project in UIBE (NO. 11YBGLX03). The second author is supported by the National Natural Science Foundation of China (No. 11101096).

References

[1] H. Brezis, J.L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Comp. Madrid 10 (1997) 443–469.
[2] J.L. Vazquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal. 173 (2000) 103–153.
[3] S. Filippas, A. Tertikas, Optimizing improved Hardy inequalities, J. Funct. Anal. 192 (1) (2002) 186–233. Corrigendum; J. Funct. Anal. 255 (2008) 2095.
[4] Adimurthi, S. Filippas, A. Tertikas, On the best constant of Hardy–Sobolev inequalities, Nonlinear Anal. 70 (2009) 2826–2833.
[5] L. D'Ambrosio, Hardy inequalities related to Grushin type operators, Proc. Amer. Math. Soc. 132 (2004) 725–734.
[6] W. Beckner, On the Grushin operator and hyperbolic symmetry, Proc. Amer. Math. Soc. 129 (2001) 1233–1246.
[7] R. Monti, D. Morbidelli, Kelvin transform for Grushin operators and critical semilinear equations, Duke Math. J. 131 (2006) 167–202.
[8] B. Franchi, C.E. Gutiérrez, R.L. Wheeden, Weighted Sobolev–Poincaré inequalities for Grushin type operators, Comm. Partial Differ. Equ. 19 (1994) 523–604.
[9] G.B. Folland, E.M. Stein, Hardy Spaces on Homogeneous Groups, Princeton University Press, Princeton, NJ, 1982.
[10] Z.M. Balogh, J.T. Tyson, Polar coordinates in the Carnot groups, Math. Z. 241 (2002) 697–730.
[11] W. Cohn, G. Lu, Best constants for Moser–Trudinger inequalities on the Heisenberg group, Indiana Univ. Math. J. 50 (2001) 1567–1591.